A $\hbar$-deformation of the W_N algebra and its vertex operators

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Abstract

In this paper,we derive a $\hbar$-deformation of the W_N algebra and its quantum Miura tranformation.The vertex operators for this $\hbar$-deformed W_N algebra and its commutation relations are also obtained.

1 Introduction

Recently,the studies of q-deformation of some infinite dimensional algebra—q-deformed affine algebra[4,6,11] ,q-deformed virasoro[2,21] algebra and W_N-algebra [2,3,9,10],have attracted much attention of physicist and mathematician. The q-deformed affine algebra and its vertex operators provide a powerful method to study the state space and the correlation function of solvable lattice model both in the bulk [16] case and the boundary case[14].However,the symmetry of q-deformed affine algebra only correspond to the current algebra (affine Lie algebra) symmetry in the conformal field theory (CFT),not to the Virasoro and W algebra type symmetry.The q-deformation of Virasoro and W algebra ,which would play the role of symmetry algebra for the solvable lattice model,has been expected for a long time. H.Awata et al also constructed the q-deformed W_N algebra (including Virasoro algebra) and associated Miura transformation from studying the Macdonald symmetrical functions[2]. On the other hand,E.Frenkel and N.Reshetikin succeeded in constructing the q-deformed classical W_N algebra and corresponding Miura transformation in analysis of the $U_q(\hat{sl}_N)$ algebra at the critical level[10].Then,B.Feigin and E.Frenkel obtained the quantum version of this q-deformed classical W_N algebra,i.e the q-deformed W_N algebra[9].The q-deformed Virasoro algebra was also given by S.Lukyanov et al in studying the Bosonization for ABF model[21].The bosonization for vertex operators of q-deformed Virasoro[17,21] and W_N algebra[2,1] have been constructed.

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However,there exists another important deformation of infinite dimensional algebra,which plays an important role in the completely integrable field theories(In order to comparation with q-deformation, we call it as $\hbar$-deformation).This deformation for affine algebra was originated by Drinfeld in studies of Yangian[7].It has been shown that the Yangian $(DY(\hat{sl}_2))$ is the dynamical non-abelian symmetry algebra for SU(2)-invariant Thirring model[13,18,19,23].Naively, the $\hbar$-deformed affine algebra (or Yangian) would play the same role in the integrable field theories as the q-deformed affine algebra in solvable lattice model.Naturally,the $\hbar$-deformed Virasoro and W algebra,which would play the role of symmetry algebra of some integrable field model,are expected.We have succeeded in constructed $\hbar$-deformed Virasoro algebra in Ref.[12] and shown that this $\hbar$-deformed Virasoro algebra is the dynamical symmetry algebra of the Restriected sine-Gordon model.In this paper,we construct the $\hbar$-deformed W_N alegbra (including the $\hbar$-deformed Virasoro algebra as a special case of N=2) , corresponding quantum Miura transformation and its vertex operators . The $\hbar$-deformed W_N algebra will become the usual no-deformed W_N algebra[8] with some center charge which is related to parameter ξ ,when $\hbar \longrightarrow 0$ and ξ and β are fixed.

This paper is arranged as follows: In section 2,we define the $\hbar$-deformed W_N algebra and its Miura transformation.The screening currents and vertex operators are derived in section 3 and 4.

2 $\hbar$-deformation of W_N algebra

In this section,we start with defining a $\hbar$-deformed W_N algebra by quantum Miura transformation.

2.1 $A^{(1)}_{N-1}$ type weight

In this subsection, we shall give some notation about $A^{(1)}_{N-1}$ type weight which will be used in the following parts of this paper. Let $\epsilon_\mu (1 \leq \mu \leq N)$ be the orthonormal basis in $\mathbf{R}^N$,which is supplied with the inner product $<\epsilon_\mu, \epsilon_\nu>= \delta_{\mu\nu}$. Set

$$\overline{\epsilon}_\mu = \epsilon_\mu - \epsilon \qquad \epsilon = \frac{1}{N}\sum_{\mu=1}^{N} \epsilon_\mu \tag{1}$$

The $A^{(1)}_{N-1}$ type weight lattice is the linear space of the

$$P = \sum_{\mu=1}^{N} Z\overline{\epsilon}_\mu$$

Note that $\sum_{\mu=1}^{N} \overline{\epsilon}_\mu = 0$.Let $\omega_\mu (1 \leq \mu \leq N-1)$ be the fundamental weights:

$$\omega_\mu = \sum_{\nu=1}^{\mu} \overline{\epsilon}_\nu$$

and the simple roots $\alpha_\mu (1 \leq \mu \leq N-1)$:

$$\alpha_\mu = \bar{\epsilon}_\mu - \bar{\epsilon}_{\mu+1} = \epsilon_\mu - \epsilon_{\mu+1} \tag{2}$$

An ordered pair $(b,a) \in \mathbf{P}^2$ is called admissible if only if there exists $\mu \in (1 \leq \mu \leq N-1)$ such that

$$b - a = \bar{\epsilon}_\mu$$

An ordered set of four weight $\begin{pmatrix} c & d \\ b & a \end{pmatrix} \in \mathbf{P}^4$ is called an admissible configuration around a face if and only if the pairs (b,a),(c,b),(d,a) and (c,d) are all admissible pairs.To each admissible configuration around a face we shall associate a Boltzmann weight in section 4.

2.2 Quantum Miura transformation

Let us consider free bosons $\lambda_i(t) \ \ (i = 1, ..., N)$ with continuous parameter $t \in \{\mathbf{R} - 0)\}$ which satisfy

$$[\lambda_i(t), \lambda_i(t')] = \frac{4sh\frac{(N-1)\hbar t}{2} sh\frac{\hbar \xi t}{2} sh\frac{\hbar(\xi+1)t}{2}}{tsh\frac{N\hbar t}{2}} \delta(t+t') \tag{3}$$

$$[\lambda_i(t), \lambda_j(t')] = \frac{4sh\frac{\hbar t}{2} sh\frac{\hbar \xi t}{2} sh\frac{\hbar(\xi+1)t}{2} e^{\mathrm{sign}(j-i)\frac{N\hbar t}{2}}}{tsh\frac{N\hbar t}{2}} \delta(t+t') \quad i \neq j \tag{4}$$

with the deformed parameter $\hbar$ and a generic parameter ξ and $\lambda_i(t)$ should be subject to the following condition

$$\sum_{l=1}^{N} \lambda_l(t) e^{l\hbar t} = 0 \tag{5}$$

One can check that the restricted condition would be compatable with Eq.(3) and Eq.(4) .

Remark: The free bosons with continuous parameter in the case of N=2 ,was first introduced by M.Jimbo et al in studying Massless XXZ mode[15].This kind bosons could be used to construct the bosonization of Yangian double with center $DY(\hat{sl}_N)$.

Let us define the fundamental operators $\Lambda_i(\beta)$ and the $\hbar$-deformed W_N algebra generators $T_i(\beta)$ for i=1,...,N as follows

$$\Lambda_i(\beta) =: exp\{-\int_{-\infty}^{\infty} \lambda_i(t) e^{i\beta t} dt\} : \tag{6}$$

$$T_l(\beta) = \sum_{1 \leq j_1 < j_2 ... < j_l \leq N} : \Lambda_{j_1}(\beta + i\frac{l-1}{2}\hbar) \Lambda_{j_2}(\beta + i\frac{l-3}{2}\hbar) .... \Lambda_{j_l}(\beta - i\frac{l-1}{2}\hbar) : \tag{7}$$

and $T_0(\beta) = 1$. Here $: O :$ stands for the usual bosonic normal ordering of some operator O such that the bosons $\lambda_i(t)$ with non-negative mode $t > 0$ are in the right. The restricted condition for bosons $\lambda_i(t)$ in

Eq.(5) results in $T_N(\beta)=1$.Actually,the generators $T_i(\beta)$ are obtained by the following quantum Miura transformation

$$
\begin{aligned}
&: (e^{i\hbar\partial_\beta}-\Lambda_1(\beta))(e^{i\hbar\partial_\beta}-\Lambda_2(\beta-i\hbar))....(e^{i\hbar\partial_\beta}-\Lambda_N(\beta-i(N-1)\hbar)) : \\
&= \sum_{l=0}^{N}(-1)^l T_l(\beta-i\frac{l-1}{2}\hbar)e^{i(N-l)\hbar\partial_\beta} \qquad (8)
\end{aligned}
$$

Remark: $e^{i\hbar\partial_\beta}$ is the $\hbar$-shift operator such that

$$e^{i\hbar\partial_\beta}f(\beta)=f(\beta+i\hbar)$$

If we take the limit of $\xi \longrightarrow -1$, the above generators $T_l(\beta)$ reduce to the classical version of $\hbar$-deformed W_N algebra,whcih can be obtained by studying the Yangian Double with center $DY(\hat{sl}_N)$ at the critical level (i.e l=-2).For the case of N=2, the corresponding classical $\hbar$-deformed W_2 (Virasoro) algebra has been given by Hou et al [5].Thus,we call the limit ($\xi \longrightarrow -1$ with $\hbar$,β fixed) as the classical limit.

Let us consider another limit:$\hbar \longrightarrow 0$ with fixed ξ . Then we have :$\Lambda_i(\beta)=1+i\hbar\chi_i(\beta)+o(\hbar)$ and $e^{i\hbar\partial_\beta}=1+i\hbar\partial_\beta+o(\hbar)$. Hence the right hand side of (8) in this limit becomes

$$: (i\hbar)^N(\partial_\beta-\chi_1(\beta))(\partial_\beta-\chi_2(\beta)).....(\partial_\beta-\chi_N(\beta)) : +o(\hbar^N) \qquad (9)$$

and we obtain the normally ordered Miura transformation corresponding to the non-deformed W_N algebra introduced by V.Fateev and S.Lukyanov[8].Therefore,the no-deformed W_N algebra (the ordinary one) with the center charge $(N-1)-\frac{N(N+1)}{\xi(1+\xi)}$ can be obtained by taking this kind limit.In this sense, we call this limit ($\hbar \longrightarrow 0$ with fixed ξ and β) as the conformal limit.

2.3 Relations of $\hbar$-deformed W_N algebra

In order to get the commuting relations for bosonic operators, we should give a comment about regularization :When one compute the exchange relation of bosonic operators ,one often encounter an integral as follows

$$\int_0^\infty F(t)dt$$

which is divergent at $t=0$ .Hence we adopt the regularization given by Jimbo et al[15]. Namely, the above integral should be understood as the contour integral

$$\int_C F(t)\frac{log(-t)}{2i\pi}dt \qquad (10)$$

where the contour C is chosen as the same as in the Ref.[15]. From the definition of fundamental operators $\Lambda_i(\beta)$ and the the commutation relations of bosons $\lambda_i(t)$,we can derive the following OPEs

$$\Lambda_i(\beta_1)\Lambda_i(\beta_2) = \phi_{i=i}(\beta_2-\beta_1) : \Lambda_i(\beta_1)\Lambda_i(\beta_2) : \tag{11}$$

$$\Lambda_i(\beta_1)\Lambda_j(\beta_2) = \phi_{i<j}(\beta_2-\beta_1) : \Lambda_i(\beta_1)\Lambda_i(\beta_2) : \quad i<j \tag{12}$$

$$\Lambda_i(\beta_1)\Lambda_j(\beta_2) = \phi_{i>j}(\beta_2-\beta_1) : \Lambda_i(\beta_1)\Lambda_j(\beta_2) : \quad i>j \tag{13}$$

$$\begin{aligned}
\phi_{i=i}(\beta) &= \frac{\Gamma(\frac{i\beta}{N\hbar}-\frac{\xi}{N})\Gamma(\frac{i\beta}{N\hbar}+1-\frac{1}{N})\Gamma(\frac{i\beta}{N\hbar}+\frac{1+\xi}{N})\Gamma(\frac{i\beta}{N\hbar}+1)}{\Gamma(\frac{i\beta}{N\hbar})\Gamma(\frac{i\beta}{N\hbar}-\frac{1+\xi}{N}+1)\Gamma(\frac{i\beta}{N\hbar}+\frac{1}{N})\Gamma(\frac{i\beta}{N\hbar}+1+\frac{\xi}{N})}\\
\phi_{i<j}(\beta) &= \frac{\Gamma(\frac{i\beta}{N\hbar}-\frac{1}{N})\Gamma(\frac{i\beta}{N\hbar}-\frac{\xi}{N})\Gamma(\frac{i\beta}{N\hbar}+\frac{1+\xi}{N})}{\Gamma(\frac{i\beta}{N\hbar}-\frac{1+\xi}{N})\Gamma(\frac{i\beta}{N\hbar}+\frac{\xi}{N})\Gamma(\frac{i\beta}{N\hbar}+\frac{1}{N})}\\
\phi_{i>j}(\beta) &= \frac{\Gamma(\frac{i\beta}{N\hbar}+1-\frac{1}{N})\Gamma(\frac{i\beta}{N\hbar}+1-\frac{\xi}{N})\Gamma(\frac{i\beta}{N\hbar}+1+\frac{1+\xi}{N})}{\Gamma(\frac{i\beta}{N\hbar}+1-\frac{1+\xi}{N})\Gamma(\frac{i\beta}{N\hbar}+1+\frac{\xi}{N})\Gamma(\frac{i\beta}{N\hbar}+1+\frac{1}{N})}
\end{aligned} \tag{14}$$

To caculate the general OPEs,the integral representation for Γ-function is very useful

$$\Gamma(z) = exp\{\int_0^\infty (\frac{e^{-zt}-e^{-t}}{1-e^{-t}}+(z-1)e^{-t})\frac{dt}{t}\} \quad , \quad Re(z)>0 \tag{15}$$

Remark: The above OPEs can be considered as the operator scaling limit of q-deformed $\Lambda_i(z)$ given by B.Feigin and E.Frenkel in studying the q-deformed W_N algebra [9](the very similar $\Lambda_i(z)$ was also constructed by H.Awata et al [2]).The scaling limit is taken as following way

$$z = p^{\frac{-i\beta}{\hbar}} \quad , \quad q = p^{-\xi} \quad , \quad \Lambda_i(\beta) = \lim_{p\longrightarrow 1}\Lambda_i(z) \equiv \lim_{p\longrightarrow 1}\Lambda_i(p^{\frac{-i\beta}{\hbar}}) \tag{16}$$

Theorem 1 The generators $T_1(\beta)$ and $T_m(\beta)$ of $\hbar$-deformed W_N algebra satisfy the following relations

$$\begin{aligned}
&f_{1m}^{-1}(\beta_2-\beta_1)T_1(\beta_1)T_m(\beta_2) - f_{1m}^{-1}(\beta_1-\beta_2)T_m(\beta_2)T_1(\beta_1) = -2i\pi\{i\hbar\xi(\xi+1)\\
&\times(T_{m+1}(\beta_2+\frac{i\hbar}{2})\delta(\beta_1-\beta_2-i\frac{m+1}{2}\hbar) - T_{m+1}(\beta_2-\frac{i\hbar}{2})\delta(\beta_1-\beta_2+i\frac{m+1}{2}\hbar))\}
\end{aligned} \tag{17}$$

where

$$f_{1m}(\beta) = \frac{\Gamma(\frac{i\beta}{N\hbar}+1-\frac{1+m}{2N})\Gamma(\frac{i\beta}{N\hbar}+1+\frac{1-m}{2N})\Gamma(\frac{i\beta}{N\hbar}-\frac{\xi}{N}-\frac{1-m}{2N})\Gamma(\frac{i\beta}{N\hbar}+\frac{\xi}{N}+\frac{1+m}{2N})}{\Gamma(\frac{i\beta}{N\hbar}+1-\frac{\xi}{N}-\frac{1+m}{2N})\Gamma(\frac{i\beta}{N\hbar}+1+\frac{\xi}{N}+\frac{1-m}{2N})\Gamma(\frac{i\beta}{N\hbar}-\frac{1-m}{2N})\Gamma(\frac{i\beta}{N\hbar}+\frac{1+m}{2N})} \tag{18}$$

Proof Using the OPEs Eq.(11)-Eq.(13), we obtain that when Imβ_2 <<Imβ_1

$$\Lambda_l(\beta_1) : \Lambda_{j_1}(\beta_2+i\frac{m-1}{2}\hbar)....\Lambda_{j_m}(\beta_2-i\frac{m-1}{2}\hbar) :$$

is equal to

$$f_{1m}(\beta_2-\beta_1) : \Lambda_l(\beta_1)\Lambda_{j_1}(\beta_2+i\frac{m-1}{2}\hbar)....\Lambda_{j_m}(\beta_2-i\frac{m-1}{2}\hbar) :$$

if $l = j_k$ for some $k \in \{1, ..., m\}$; and

$$f_{1m}(\beta_2-\beta_1)\frac{(i\frac{\beta_2-\beta_1}{N\hbar}-\frac{\xi}{N}-\frac{1}{2N}+\frac{2k-m}{2N})(i\frac{\beta_2-\beta_1}{N\hbar}+\frac{\xi}{N}+\frac{1}{2N}+\frac{2k-m}{2N})}{(i\frac{\beta_2-\beta_1}{N\hbar}-\frac{1}{2N}+\frac{2k-m}{2N})(i\frac{\beta_2-\beta_1}{N\hbar}+\frac{1}{2N}+\frac{2k-m}{2N})} : \Lambda_l(\beta_1)$$
$$\times \Lambda_{j_1}(\beta_2+i\frac{m-1}{2}\hbar)....\Lambda_{j_m}(\beta_2-i\frac{m-1}{2}\hbar) :$$

if $j_k < l < j_{k+1}$. Here and below the case $l < j_1$ corresponds to $k = 0$ and the case $l > j_m$ corresponds to $k = m$. On the other hand,when $\mathrm{Im}\beta_2 >>\mathrm{Im}\beta_1$,

$$: \Lambda_{j_1}(\beta_2+i\frac{m-1}{2}\hbar)....\Lambda_{j_m}(\beta_2-i\frac{m-1}{2}\hbar) : \Lambda_l(\beta_1)$$

is equal to

$$f_{1m}(\beta_1-\beta_2) : \Lambda_{j_1}(\beta_2+i\frac{m-1}{2}\hbar)....\Lambda_{j_m}(\beta_2-i\frac{m-1}{2}\hbar)\Lambda_l(\beta_1) :$$

if $l = j_k$ for some $k \in \{1, ..., m\}$; and

$$f_{1m}(\beta_1-\beta_2)\frac{(i\frac{\beta_1-\beta_2}{N\hbar}-\frac{\xi}{N}-\frac{1}{2N}-\frac{2k-m}{2N})(i\frac{\beta_1-\beta_2}{N\hbar}+\frac{\xi}{N}+\frac{1}{2N}-\frac{2k-m}{2N})}{(i\frac{\beta_1-\beta_2}{N\hbar}-\frac{1}{2N}-\frac{2k-m}{2N})(i\frac{\beta_1-\beta_2}{N\hbar}+\frac{1}{2N}-\frac{2k-m}{2N})} : \Lambda_{j_1}(\beta_2+i\frac{m-1}{2}\hbar)$$
$$\times ....\Lambda_{j_m}(\beta_2-i\frac{m-1}{2}\hbar)\Lambda_l(\beta_1) :$$

if $j_k < l < j_{k+1}$.Noting that

$$\lim_{\epsilon\longrightarrow 0+}(\frac{1}{x+i\epsilon}-\frac{1}{x-i\epsilon}) = -2i\pi\delta(x) \tag{19}$$

we can obtain the commutation relations Eq.(17) for $T_1(\beta_1)$ and $T_m(\beta_2)$ after some straightforward caculation.Therefore, we complete the proof of Theorem 1 $\Box$

In fact, the commutation relations for the generators of $\hbar$-deformed W_N algebra have already been defined from the commutation relations of fundamental operators $\Lambda_i(\beta)$ in Eq.(6) and the corresponding quantum Miura transformation in Eq.(7). So, one can also derive some similar commutation relations between $T_i(\beta)$ and $T_j(\beta)$ with $i, j > 1$ using the same method as that in proof of the Theorem 1. These commutation relations are quadratic, and involve products of $T_{i-r}(\beta)$ and $T_{j+r}(\beta)$ with $r = 1, ..., \min(i, j) - 1$

In the case of N=2 ,this $\hbar$-deformed W_2 algebra becomes $\hbar$-deformed Virasoro algebra,which has been studied by us in the Ref.[12]. Here,we give an example for the case $N = 3$.The generators of this case are

$$T_1(\beta) = \Lambda_1(\beta) + \Lambda_2(\beta) + \Lambda_3(\beta) \tag{20}$$
$$T_2(\beta) =: \Lambda_1(\beta+\frac{i\hbar}{2})\Lambda_2(\beta-\frac{i\hbar}{2}) : + : \Lambda_1(\beta+\frac{i\hbar}{2})\Lambda_3(\beta-\frac{i\hbar}{2}) : + : \Lambda_2(\beta+\frac{i\hbar}{2})\Lambda_3(\beta-\frac{i\hbar}{2}) : \tag{21}$$

The commutation relations for these two generators are

$$\begin{aligned}
&f_{11}^{-1}(\beta_2-\beta_1)T_1(\beta_1)T_1(\beta_2)-f_{11}^{-1}(\beta_1-\beta_2)T_1(\beta_2)T_1(\beta_1)=-2i\pi\{i\hbar\xi(\xi+1)\\
&\quad\times(T_2(\beta_2+\frac{i\hbar}{2})\delta(\beta_1-\beta_2-i\hbar)-T_2(\beta_2-\frac{i\hbar}{2})\delta(\beta_1-\beta_2+i\hbar))\} \qquad (22)\\
&f_{12}^{-1}(\beta_2-\beta_1)T_1(\beta_1)T_2(\beta_2)-f_{12}^{-1}(\beta_1-\beta_2)T_2(\beta_2)T_1(\beta_1)=-2i\pi\{i\hbar\xi(\xi+1)\\
&\quad\times(\delta(\beta_1-\beta_2-i\frac{3}{2}\hbar)-\delta(\beta_1-\beta_2+i\frac{3}{2}\hbar))\} \qquad (23)\\
&f_{22}^{-1}(\beta_2-\beta_1)T_2(\beta_1)T_2(\beta_2)-f_{22}^{-1}(\beta_1-\beta_2)T_2(\beta_2)T_2(\beta_1)=-2i\pi\{i\hbar\xi(\xi+1)\\
&\quad\times(T_1(\beta_2+\frac{i\hbar}{2})\delta(\beta_1-\beta_2-i\hbar)-T_1(\beta_2-\frac{i\hbar}{2})\delta(\beta_1-\beta_2+i\hbar))\} \qquad (24)
\end{aligned}$$

where the coefficient function $f_{ij}(\beta)$ are

$$\begin{aligned}
f_{11}(\beta)&=\frac{\Gamma(\frac{i\beta}{N\hbar}+1-\frac{1}{N})\Gamma(\frac{i\beta}{N\hbar}+1)\Gamma(\frac{i\beta}{N\hbar}-\frac{\xi}{N})\Gamma(\frac{i\beta}{N\hbar}+\frac{\xi}{N}+\frac{1}{N})}{\Gamma(\frac{i\beta}{N\hbar}+1-\frac{\xi}{N}-\frac{1}{N})\Gamma(\frac{i\beta}{N\hbar}+1+\frac{\xi}{N})\Gamma(\frac{i\beta}{N\hbar})\Gamma(\frac{i\beta}{N\hbar}+\frac{1}{N})}\\
f_{12}(\beta)&=\frac{\Gamma(\frac{i\beta}{N\hbar}+1-\frac{3}{2N})\Gamma(\frac{i\beta}{N\hbar}+1-\frac{1}{2N})\Gamma(\frac{i\beta}{N\hbar}-\frac{\xi}{N}+\frac{1}{2N})\Gamma(\frac{i\beta}{N\hbar}+\frac{\xi}{N}+\frac{3}{2N})}{\Gamma(\frac{i\beta}{N\hbar}+1-\frac{\xi}{N}-\frac{3}{2N})\Gamma(\frac{i\beta}{N\hbar}+1+\frac{\xi}{N}-\frac{1}{2N})\Gamma(\frac{i\beta}{N\hbar}+\frac{1}{2N})\Gamma(\frac{i\beta}{N\hbar}+\frac{3}{2N})}\\
f_{11}(\beta)&=f_{22}(\beta)
\end{aligned}$$

3 Screening currents

In this section,we will consider the screening currents for the $\hbar$-deformed W_N algebra.First,we introduce some zero mode operators. To each vector $\alpha\in\mathbf{P}$ (the $A_{N-1}^{(1)}$ type weight lattice defined in section 2.1),we associate operators P_α, Q_α which satisfy

$$[iP_\alpha,Q_\beta]=<\alpha,\beta> \quad , \quad (\alpha,\beta\in\mathbf{P}) \qquad (25)$$

We shall deal with the bosonic Fock spaces $F_{l,k}$ $(l,k\in\mathbf{P})$ generated by $\lambda_i(-t)$ $(t>0)$ over the vacuum states $|l,k>$ .The vacuum states $|l,k>$ are defined by

$$\begin{aligned}
&\lambda_i(t)|l,k>=0 \qquad \text{if} \quad t>0\\
&P_\beta|l,k>=<\beta,\alpha_+l+\alpha_-k>|l,k>\\
&|l,k>=e^{i\alpha_+Q_l+i\alpha_-Q_k}|0,0>
\end{aligned}$$

where $\alpha_\pm$ are some parameters related to ξ

$$\alpha_+=-\sqrt{\frac{1+\xi}{\xi}} \quad , \quad \alpha_-=\sqrt{\frac{\xi}{1+\xi}} \qquad (26)$$

and we also introduce α_0

$$\alpha_0=\sqrt{\xi(1+\xi)} \qquad (27)$$

To each simple root α_j $(j = 1, ..., N-1)$, let us introduce two series bosons $s_j^{\pm}(t)$ which are defined by

$$s_j^+(t) = \frac{e^{j\frac{\hbar}{2}t}}{2sh\frac{\xi\hbar t}{2}}(\lambda_j(t) - \lambda_{j+1}(t)) \tag{28}$$

$$s_j^-(t) = \frac{e^{j\frac{\hbar}{2}t}}{2sh\frac{(1+\xi)\hbar t}{2}}(\lambda_j(t) - \lambda_{j+1}(t)) \tag{29}$$

By these simple root bosons, we can define the screening currents as follows

$$S_j^+(\beta) =: exp\{-\int_{-\infty}^{\infty} s_j^+(t)e^{i\beta t}dt\} : e^{-i\alpha_+ Q_{\alpha_j}} \tag{30}$$

$$S_j^-(\beta) =: exp\{\int_{-\infty}^{\infty} s_j^-(t)e^{i\beta t}dt\} : e^{-i\alpha_- Q_{\alpha_j}} \tag{31}$$

Then we have

Theorem 2 The screening currents $S_j^+(\beta)$ satisfy

$$\begin{aligned}
&[: (e^{i\hbar\partial_\beta} - \Lambda_1(\beta))(e^{i\hbar\partial_\beta} - \Lambda_2(\beta - i\hbar))....(e^{i\hbar\partial_\beta} - \Lambda_N(\beta - i(N-1)\hbar)) :, S_j^+(\sigma)] \\
&= i\hbar(1+\xi)\{: (e^{i\hbar\partial_\beta} - \Lambda_1(\beta))....(e^{i\hbar\partial_\beta} - \Lambda_{j-1}(\beta - i(j-2)\hbar)) \\
&\quad \times D_{\sigma, i\hbar\xi}(2\pi i\delta(\sigma - \beta - i\frac{j+\xi}{2}\hbar)A_j^+(\sigma)) \\
&\quad \times e^{i\hbar\partial_\beta}(e^{i\hbar\partial_\beta} - \Lambda_{j+2}(\beta - i(j+1)\hbar)).....(e^{i\hbar\partial_\beta} - \Lambda_N(\beta - i(N-1)\hbar)) :
\end{aligned} \tag{32}$$

with

$$A_j^+(\sigma) =: \Lambda_j(\sigma - i\frac{j+\xi}{2}\hbar)S_j^+(\sigma) :$$

and the operator $D_{\sigma, i\hbar\xi}$ is a diffenece operator with variable σ

$$D_{\sigma,\eta}f(\sigma) \equiv f(\sigma) - f(\sigma + \eta)$$

Proof From the formulas Eq.(28) ,we obtain the following commutation relations

$$[\lambda_j(t), s_j^+(t')] = -\frac{2e^{\frac{t(1-j)}{2}\hbar}sh\frac{(1+\xi)\hbar t}{2}}{t}\delta(t+t')$$

$$[\lambda_{j+1}(t), s_j^+(t')] = \frac{2e^{-\frac{t(1+j)}{2}\hbar}sh\frac{(1+\xi)\hbar t}{2}}{t}\delta(t+t')$$

$$[\lambda_j(t), s_l^+(t')] = 0 \quad , \quad \text{if} \quad |j-l| > 1$$

From these commutation relations and the formula Eq.(15) ,we can derive the following OPEs

$$\Lambda_j(\beta_1)S_j^+(\beta_2) = f_{jj}^+(\beta_2 - \beta_1) : \Lambda_j(\beta_1)S_j^+(\beta_2) : \tag{33}$$

$$S_j^+(\beta_1)\Lambda_j(\beta_2) = f_{jj}^+(\beta_2 - \beta_1) : S_j^+(\beta_1)\Lambda_j(\beta_2) : \tag{34}$$

$$\Lambda_{j+1}(\beta_1)S_j^+(\beta_2) = f_{j+1\ \ j}^+(\beta_2 - \beta_1) : \Lambda_{j+1}(\beta_1)S_j^+(\beta_2) : \tag{35}$$

$$S_j^+(\beta_1)\Lambda_{j+1}(\beta_2) = f_{j+1\ \ j}^+(\beta_2 - \beta_1) : S_j^+(\beta_1)\Lambda_{j+1}(\beta_2) : \tag{36}$$

$$S_j^+(\beta_1)\Lambda_l(\beta_2) =: S_j^+(\beta_1)\Lambda_l(\beta_2) :=: \Lambda_l(\beta_2)S_j^+(\beta_1) : \quad , \quad \text{if} \quad |j-l| > 1 \tag{37}$$

and

$$f^{+}_{jj}(\beta)=\frac{\frac{i\beta}{N\hbar}-\frac{\xi}{2N}-\frac{1}{N}+\frac{j}{2N}}{\frac{i\beta}{N\hbar}-\frac{\xi}{2N}+\frac{j}{2N}+\frac{\xi}{N}}$$
$$f^{+}_{j+1\ \ j}(\beta)=\frac{\frac{i\beta}{N\hbar}-\frac{\xi}{2N}+\frac{1+\xi}{N}+\frac{j}{2N}}{\frac{i\beta}{N\hbar}-\frac{\xi}{2N}+\frac{j}{2N}}$$

The formula Eq.(37) means that

$$\begin{aligned}\text{LHS of Eq.(32)} = \quad & :(e^{i\hbar\partial_\beta}-\Lambda_1(\beta))....(e^{i\hbar\partial_\beta}-\Lambda_{j-1}(\beta-i(j-2)\hbar))\\ &\times[(e^{i\hbar\partial_\beta}-\Lambda_j(\beta-i(j-1)\hbar))(e^{i\hbar\partial_\beta}-\Lambda_{j+1}(\beta-i(j)\hbar)),S^{+}_{j}(\sigma)]\\ &\times(e^{i\hbar\partial_\beta}-\Lambda_{j+2}(\beta-i(j+1)\hbar)).....(e^{i\hbar\partial_\beta}-\Lambda_N(\beta-i(N-1)\hbar)): \qquad (38)\end{aligned}$$

Therefore, it is sufficient to consider the commutation relation

$$[:(e^{i\hbar\partial_\beta}-\Lambda_j(\beta-i(j-1)\hbar))(e^{i\hbar\partial_\beta}-\Lambda_{j+1}(\beta-i(j)\hbar)):,S^{+}_{j}(\sigma)]$$

According to the OPEs Eq.(33)—Eq.(36) ,we can derive that

$$[:\Lambda_j(\beta-i(j-1)\hbar)\Lambda_{j+1}(\beta-i(j)\hbar):,S^{+}_{j}(\sigma)]=0$$

Now we only need to consider the commutation relations between the term $\Lambda_j(\beta-i(j-1)\hbar)+\Lambda_{j+1}(\beta-i(j-1)\hbar)$ and the screening current $S^{+}_{j}(\sigma)$. From the OPEs Eq.(33)—Eq.(36) , the formula Eq.(19), noting that

$$:\Lambda_j(\beta-i\frac{j-\xi}{2}\hbar)S^{+}_{j}(\beta+i\xi\hbar):=:\Lambda_{j+1}(\beta-i\frac{j-\xi}{2}\hbar)S^{+}_{j}(\beta):$$

and using the same method as that in the proof of Theorem 1, we have the following commutation relation

$$[\Lambda_j(\beta-i(j-1)\hbar)+\Lambda_{j+1}(\beta-i(j-1)\hbar),S^{+}_{j}(\sigma)]=i\hbar(1+\xi)D_{\sigma,i\hbar\xi}(2\pi i\delta(\sigma-\beta-i\frac{j+\xi}{2}\hbar):A^{+}_{j}(\sigma)):$$

Therefore, the Eq.(32) has been obtained. □

Using the same method, we can have

Theorem 3 The second series screening currents $S^{-}_{j}(\sigma)$ satisfy

$$\begin{aligned}&[:(e^{i\hbar\partial_\beta}-\Lambda_1(\beta))(e^{i\hbar\partial_\beta}-\Lambda_2(\beta-i\hbar))....(e^{i\hbar\partial_\beta}-\Lambda_N(\beta-i(N-1)\hbar)):,S^{-}_{j}(\sigma)]\\ &=i\hbar\xi\{:(e^{i\hbar\partial_\beta}-\Lambda_1(\beta))....(e^{i\hbar\partial_\beta}-\Lambda_{j-1}(\beta-i(j-2)\hbar))\\ &\quad\times D_{\sigma,-i\hbar(1+\xi)}(2\pi i\delta(\sigma-\beta+i\frac{-j+\xi+1}{2}\hbar)A^{-}_{j}(\sigma))\\ &\quad\times e^{i\hbar\partial_\beta}(e^{i\hbar\partial_\beta}-\Lambda_{j+2}(\beta-i(j+1)\hbar)).....(e^{i\hbar\partial_\beta}-\Lambda_N(\beta-i(N-1)\hbar)): \qquad (39)\end{aligned}$$

with

$$A_j^-(\sigma) =: \Lambda_j(\sigma + i\frac{-j+\xi+1}{2}\hbar)S_j^-(\sigma):$$

Therefore, the screening currents $S_j^{\pm}(\beta)$ commute with any $\hbar$-deformed W_N algebra generators up to total difference.

Remark: Let us take the conformal limit ($\hbar \longrightarrow 0$ and with ξ and β fixed) ,the screening currents $S_j^{\pm}(\beta)$ will become the ordinary screening current [8].

Theorem 2 and Theorem 3 imply that one can construct the intertwining operators (namely, vertex operators) for the $\hbar$-deformed W_N algebra using the screening currents $S_j^{\pm}(\beta)$.In the next section we shall construct the vertex operators for the $\hbar$-deformed W_N algebra.

4 The vertex operators and its exchange relations

In this section, we construct the type I and type II vertex operators for this $\hbar$-deformed W_N algebra through the two series screening currents $S_j^{\pm}(\beta)$ .Firstly, we set

$$\hat{\pi}_\mu = \alpha_0 P_{\bar{\epsilon}_\mu} \quad , \quad \hat{\pi}_{\mu\nu} = \hat{\pi}_\mu - \hat{\pi}_\nu \tag{40}$$

and

$$\hat{\pi}_{\mu\nu}F_{l,k} =< \epsilon_\mu - \epsilon_\nu, -(1+\xi)l + \xi k > F_{l,k} \tag{41}$$

Note that

$$e^{-i\alpha_\pm Q_\gamma}\hat{\pi}_\sigma e^{i\alpha_\pm Q_\gamma} = \hat{\pi}_\sigma + \alpha_0\alpha_\pm < \sigma, \gamma > \tag{42}$$

and this formula is very useful for caculating commutation relations of vertex operators. To each fundamental weight of ω_j $(j = 1, ..., N-1)$,let us introduce two series bosons $a_j(t)$ and $a'_j(t)$ which are defined by

$$a_j = \sum_{k=1}^{j} \frac{e^{-\frac{\hbar(j-2k+1)t}{2}}}{2sh\frac{\hbar\xi t}{2}}\lambda_k(t) \quad , \quad a'_j = \sum_{k=1}^{j} \frac{e^{-\frac{\hbar(j-2k+1)t}{2}}}{2sh\frac{\hbar(1+\xi)t}{2}}\lambda_k(t) \tag{43}$$

and also define

$$U_{\omega_j}(\beta) =: exp\{\int_{-\infty}^{\infty} a_j(t)e^{i\beta t}dt\}: e^{i\alpha_+ Q_{\omega_j}} \quad , \quad U'_{\omega_j}(\beta) =: exp\{\int_{-\infty}^{\infty} -a'_j(t)e^{i\beta t}dt\}: e^{i\alpha_- Q_{\omega_j}} \tag{44}$$

Because the vertex operators associated with each fundamental weight ω_j $(= 2, ...N-1)$ can be constructed from the skew-symmetric fusion of the basic ones $U_{\omega_1}(\beta)$ and $U'_{\omega_1}(\beta)$ [1],it is sufficient to only

deal with the vertex operators corresponding to the fundamental weight ω_1.In order to caculate the exchange relations of the vertex operators,we first derive the following commutation relations

$$[a_j(t), s_j^+(t')] = -\frac{sh\frac{\hbar(1+\xi)t}{2}}{tsh\frac{\hbar\xi t}{2}}\delta_{j,l}\delta(t+t') \quad , \quad [a_j(t), s_j^-(t')] = -\frac{1}{t}\delta_{j,l}\delta(t+t')$$

$$[a'_j(t), s_j^-(t')] = -\frac{sh\frac{\hbar\xi t}{2}}{tsh\frac{\hbar(1+\xi)t}{2}}\delta_{j,l}\delta(t+t') \quad , \quad [a'_j(t), s_j^+(t')] = -\frac{1}{t}\delta_{j,l}\delta(t+t')$$

$$[a_1(t), a_1(t')] = -\frac{sh\frac{(N-1)\hbar t}{2}sh\frac{(1+\xi)\hbar t}{2}}{tsh\frac{N\hbar t}{2}sh\frac{\xi\hbar t}{2}}\delta(t+t')$$

$$[a_1(t), a'_1(t')] = -\frac{sh\frac{(N-1)\hbar t}{2}}{tsh\frac{N\hbar t}{2}}\delta(t+t')$$

$$[a'_1(t), a'_1(t')] = -\frac{sh\frac{(N-1)\hbar t}{2}sh\frac{\xi\hbar t}{2}}{tsh\frac{N\hbar t}{2}sh\frac{(1+\xi)\hbar t}{2}}\delta(t+t')$$

From the above relations, and taking the regularization in section 2.3, we can derive the following exchange relation

$$\begin{aligned}
&U_{\omega_1}(\beta_1)U_{\omega_1}(\beta_2) = r_1(\beta_1-\beta_2)U_{\omega_1}(\beta_2)U_{\omega_1}(\beta_1)\\
&U'_{\omega_1}(\beta_1)U'_{\omega_1}(\beta_2) = r'_1(\beta_1-\beta_2)U'_{\omega_1}(\beta_2)U'_{\omega_1}(\beta_1)\\
&U_{\omega_1}(\beta_1)U'_{\omega_1}(\beta_2) = \tau_1(\beta_1-\beta_2)U'_{\omega_1}(\beta_2)U_{\omega_1}(\beta_1)\\
&S_j^+(\beta_1)S_{j+1}^+(\beta_2) = -f(\beta_1-\beta_2,0)S_{j+1}^+(\beta_2)S_j^+(\beta_1)\\
&S_j^-(\beta_1)S_{j+1}^-(\beta_2) = -f'(\beta_1-\beta_2,0)S_{j+1}^-(\beta_2)S_j^-(\beta_1)\\
&U_{\omega_1}(\beta_1)S_1^+(\beta_2) = -f(\beta_1-\beta_2,0)S_1^+(\beta_2)U_{\omega_1}(\beta_1)\\
&U'_{\omega_1}(\beta_1)S_1^- = -f'(\beta_1-\beta_2,0)S_1^-(\beta_2)U'_{\omega_1}(\beta_1)\\
&U_{\omega_1}(\beta_1)S_1^-(\beta_2) = -U_{\omega_1}(\beta_1)S_1^-(\beta_2) \quad , \quad U'_{\omega_1}(\beta_1)S_1^+(\beta_2) = -U'_{\omega_1}(\beta_1)S_1^+(\beta_2)
\end{aligned} \tag{45}$$

where the fundamental function $f(\beta, w)$ and $f'(\beta, w)$ (which play a very important role in constructing the vertex operators) are defined by

$$f(\beta, w) = \frac{sin\pi(\frac{i\beta}{\hbar\xi} - \frac{1}{2\xi} - \frac{w}{\xi})}{sin\pi(\frac{i\beta}{\hbar\xi} + \frac{1}{2\xi})} \quad , \quad f'(\beta, w) = \frac{sin\pi(\frac{i\beta}{\hbar(1+\xi)} + \frac{1}{2(1+\xi)} + \frac{w}{1+\xi})}{sin\pi(\frac{i\beta}{\hbar(1+\xi)} - \frac{1}{2(1+\xi)})} \tag{46}$$

and

$$r(\beta) = exp\{-\int_0^\infty \frac{2sh\frac{(N-1)\hbar t}{2}sh\frac{(1+\xi)\hbar t}{2}shi\beta t}{tsh\frac{N\hbar t}{2}sh\frac{\xi\hbar t}{2}}dt\} \tag{47}$$

$$r'(\beta) = exp\{-\int_0^\infty \frac{2sh\frac{(N-1)\hbar t}{2}sh\frac{\xi\hbar t}{2}shi\beta t}{tsh\frac{N\hbar t}{2}sh\frac{(1+\xi)\hbar t}{2}}dt\} \tag{48}$$

$$\tau(\beta) = \frac{sin\pi(\frac{1}{2N} - \frac{i\beta}{N\hbar})}{sin\pi(\frac{1}{2N} + \frac{i\beta}{N\hbar})} \tag{49}$$

Now let us define the type I vertex operators $Z'_\mu(\beta)$ and the type II vertex operators $Z_\mu(\beta)$ $(\mu=1,...N)$

$$Z_\mu(\beta)=\int_{C_1}\prod_{j=1}^{\mu-1}d\eta_j\ \ U'_{\omega_1}(\beta)S_1^-(\eta_1)S_2^-(\eta_2)...S_{\mu-1}^-(\eta_{\mu-1})\prod_{j=1}^{\mu-1}f'(\eta_j-\eta_{j-1},\hat{\pi}_{j\mu}) \tag{50}$$

$$Z'_\mu(\beta)=\int_{C_2}\prod_{j=1}^{\mu-1}d\eta_j\ \ U_{\omega_1}(\beta)S_1^+(\eta_1)S_2^+(\eta_2)...S_{\mu-1}^+(\eta_{\mu-1})\prod_{j=1}^{\mu-1}f(\eta_j-\eta_{j-1},\hat{\pi}_{j\mu}) \tag{51}$$

It is easy to see that the vertex operator $Z_\mu(\beta)$ and $Z'_\mu(\beta)$ are some bosonic operator intertwing the Fock spaces $F_{l,k}$

$$Z_\mu(\beta)\ \ :\ \ F_{l,k}\longrightarrow F_{l,k+\overline{\epsilon}_\mu}\ \ ,\ \ Z'_\mu(\beta)\ \ :\ \ F_{l,k}\longrightarrow F_{l+\overline{\epsilon}_\mu,k} \tag{52}$$

Here we set $\eta_0=\beta$, the integration contour C_1 is chosen as :the contour corresponding to the integration variable η_j enclose the poles $\eta_{j-1}+i\frac{\hbar}{2}-i\hbar\xi n(0\leq n)$, and the other integration contour C_2 is chosen as : the contour corresponding to the integration variable η_j enclose the poles $\eta_{j-1}-i\frac{\hbar}{2}-i(1+\xi)\hbar n(0\leq n)$.

The constructure form of type I (type II) vertex operators of ours seems to be similar as that of vertex operators for $A_{N-1}^{(1)}$ face model given by Y.Asai et al [1],but with different bosonic operators and "coefficient parts" function $f(\beta,w)$ $(f'(\beta,w)$ .Thus the same trick [1] can be used to caculate the commutation relations for our vertex operators. Using the method which was presented by Y.Asai et al in the apppendix B of the Ref.[1], we can derive the commutation relations for vertex operators $Z_\mu(\beta)$ and $Z'_\mu(\beta)$

$$Z'_\mu(\beta_1)Z'_\nu(\beta_2)=\sum_{\mu'\nu'}^{\overline{\epsilon}_\mu+\overline{\epsilon}_\nu=\overline{\epsilon}_{\mu'}+\overline{\epsilon}_{\nu'}}Z'_{\mu'}(\beta_2)Z'_{\nu'}(\beta_1)\hat{W}'\left(\begin{array}{ll}\hat{\pi}+\overline{\epsilon}_\mu+\overline{\epsilon}_\nu & \hat{\pi}+\overline{\epsilon}_{\nu'}\\ \hat{\pi}+\overline{\epsilon}_\nu & \hat{\pi}\end{array}|\beta_1-\beta_2\right) \tag{53}$$

$$Z_\mu(\beta_1)Z_\nu(\beta_2)=\sum_{\mu'\nu'}^{\overline{\epsilon}_\mu+\overline{\epsilon}_\nu=\overline{\epsilon}_{\mu'}+\overline{\epsilon}_{\nu'}}Z_{\mu'}(\beta_2)Z_{\nu'}(\beta_1)\hat{W}\left(\begin{array}{ll}\hat{\pi}+\overline{\epsilon}_\mu+\overline{\epsilon}_\nu & \hat{\pi}+\overline{\epsilon}_{\nu'}\\ \hat{\pi}+\overline{\epsilon}_\nu & \hat{\pi}\end{array}|\beta_1-\beta_2\right) \tag{54}$$

$$Z_\mu(\beta_1)Z'_\nu(\beta_2)=Z'_\nu(\beta_2)Z'_\mu(\beta_1)\tau(\beta_1-\beta_2) \tag{55}$$

and the braid matrices $\hat{W}\left(\begin{array}{ll}\hat{\pi}+\overline{\epsilon}_\mu+\overline{\epsilon}_\nu & \hat{\pi}+\overline{\epsilon}_{\nu'}\\ \hat{\pi}+\overline{\epsilon}_\nu & \hat{\pi}\end{array}|\beta\right)$ and $\hat{W}'\left(\begin{array}{ll}\hat{\pi}+\overline{\epsilon}_\mu+\overline{\epsilon}_\nu & \hat{\pi}+\overline{\epsilon}_{\nu'}\\ \hat{\pi}+\overline{\epsilon}_\nu & \hat{\pi}\end{array}|\beta\right)$ are some functions taking the value on operators $\hat{\pi}_{\mu\nu}$ like

$$\hat{W}'\left(\begin{array}{ll}\hat{\pi}+2\overline{\epsilon}_\mu & \hat{\pi}+\overline{\epsilon}_\mu\\ \hat{\pi}+\overline{\epsilon}_\mu & \hat{\pi}\end{array}|\beta\right)=r(\beta) \tag{56}$$

$$\hat{W}'\left(\begin{array}{ll}\hat{\pi}+\overline{\epsilon}_\mu+\overline{\epsilon}_\nu & \hat{\pi}+\overline{\epsilon}_\nu\\ \hat{\pi}+\overline{\epsilon}_\nu & \hat{\pi}\end{array}|\beta\right)=-r(\beta)\frac{sin\frac{\pi}{\xi}sin\pi(\frac{i\beta}{\hbar\xi}-\frac{\hat{\pi}_{\mu\nu}}{\xi})}{sin\pi(\frac{\hat{\pi}_{\mu\nu}}{\xi})sin\pi(\frac{i\beta}{\hbar\xi}+\frac{1}{\xi})} \tag{57}$$

$$\hat{W}'\left(\begin{array}{ll}\hat{\pi}+\overline{\epsilon}_\mu+\overline{\epsilon}_\nu & \hat{\pi}+\overline{\epsilon}_\mu\\ \hat{\pi}+\overline{\epsilon}_\nu & \hat{\pi}\end{array}|\beta\right)=r(\beta)\frac{sin\pi(\frac{i\beta}{\hbar\xi})sin\pi(\frac{1}{\xi}+\frac{\hat{\pi}_{\mu\nu}}{\xi})}{sin\pi(\frac{\hat{\pi}_{\mu\nu}}{\xi})sin\pi(\frac{i\beta}{\hbar\xi}+\frac{1}{\xi})} \tag{58}$$

$$\hat{W}\left(\begin{array}{ll}\hat{\pi}+2\overline{\epsilon}_\mu & \hat{\pi}+\overline{\epsilon}_\mu\\ \hat{\pi}+\overline{\epsilon}_\mu & \hat{\pi}\end{array}|\beta\right)=r'(\beta) \tag{59}$$

$$\hat{W}\left(\begin{array}{ll}\hat{\pi}+\overline{\epsilon}_\mu+\overline{\epsilon}_\nu & \hat{\pi}+\overline{\epsilon}_\nu \\ \hat{\pi}+\overline{\epsilon}_\nu & \hat{\pi}\end{array}|\beta\right)=-r'(\beta)\frac{sin\frac{\pi}{1+\xi}sin\pi(\frac{i\beta}{\hbar(1+\xi)}+\frac{\hat{\pi}_{\mu\nu}}{1+\xi})}{sin\pi(\frac{\hat{\pi}_{\mu\nu}}{1+\xi})sin\pi(\frac{i\beta}{\hbar(1+\xi)}-\frac{1}{1+\xi})} \tag{60}$$

$$\hat{W}\left(\begin{array}{ll}\hat{\pi}+\overline{\epsilon}_\mu+\overline{\epsilon}_\nu & \hat{\pi}+\overline{\epsilon}_\mu \\ \hat{\pi}+\overline{\epsilon}_\nu & \hat{\pi}\end{array}|\beta\right)=r'(\beta)\frac{sin\pi(\frac{i\beta}{\hbar(1+\xi)})sin\pi(\frac{1}{1+\xi}+\frac{\hat{\pi}_{\mu\nu}}{1+\xi})}{sin\pi(\frac{\hat{\pi}_{\mu\nu}}{1+\xi})sin\pi(\frac{i\beta}{\hbar(1+\xi)}-\frac{1}{1+\xi})} \tag{61}$$

Therefore, these braid matrices are not commute with the vertex operators and the exchange relations should be written in that order as Eq.(53) and Eq.(54).Noting that

$$r(-\beta)|_{\xi\longrightarrow 1+\xi}=r'(\beta)\Delta_N(\beta)$$
$$\Delta_N(\beta)=\frac{sin\pi(\frac{i\beta}{N\hbar}+\frac{1}{N})}{sin\pi(\frac{i\beta}{N\hbar}-\frac{1}{N})}$$

we find that the braid matrices $\hat{W}\left(\begin{array}{ll}\hat{\pi}+\overline{\epsilon}_\mu+\overline{\epsilon}_\nu & \hat{\pi}+\overline{\epsilon}_{\nu'} \\ \hat{\pi}+\overline{\epsilon}_\nu & \hat{\pi}\end{array}|\beta\right)$ and $\hat{W}'\left(\begin{array}{ll}\hat{\pi}+\overline{\epsilon}_\mu+\overline{\epsilon}_\nu & \hat{\pi}+\overline{\epsilon}_{\nu'} \\ \hat{\pi}+\overline{\epsilon}_\nu & \hat{\pi}\end{array}|\beta\right)$ are related to each other as follows

$$\hat{W}'\left(\begin{array}{ll}\hat{\pi}+\overline{\epsilon}_\mu+\overline{\epsilon}_\nu & \hat{\pi}+\overline{\epsilon}_{\nu'} \\ \hat{\pi}+\overline{\epsilon}_\nu & \hat{\pi}\end{array}|-\beta\right)|_{\xi\longrightarrow 1+\xi}=\Delta_N(\beta)\hat{W}\left(\begin{array}{ll}\hat{\pi}+\overline{\epsilon}_\mu+\overline{\epsilon}_\nu & \hat{\pi}+\overline{\epsilon}_{\nu'} \\ \hat{\pi}+\overline{\epsilon}_\nu & \hat{\pi}\end{array}|\beta\right)$$

When N=2,the factor $\Delta_N(\beta)=-1$,it occured in studying the $\hbar$-deformed Virasoro algebra [12]. If both sides of Eq.(53) and Eq.(54) are acted on the special Fock space $F_{l,k}$, noting that $l_{\mu\nu}$ and $k_{\mu\nu}$

$$l_{\mu\nu}=<\overline{\epsilon}_\mu-\overline{\epsilon}_\nu,l> \quad , \quad k_{\mu\nu}=<\overline{\epsilon}_\mu-\overline{\epsilon}_\nu,k>$$

are all some integer ,we have

$$Z'_\mu(\beta_1)Z'_\nu(\beta_2)|_{F_{l,k}}=\sum_{\mu'\nu'}^{\overline{\epsilon}_\mu+\overline{\epsilon}_\nu=\overline{\epsilon}_{\mu'}+\overline{\epsilon}_{\nu'}}Z'_{\mu'}(\beta_2)Z'_{\nu'}(\beta_1)|_{F_{l,k}}W'\left(\begin{array}{ll}l+\overline{\epsilon}_\mu+\overline{\epsilon}_\nu & l+\overline{\epsilon}_{\nu'} \\ l+\overline{\epsilon}_\nu & l\end{array}|\beta_1-\beta_2\right) \tag{62}$$

$$Z_\mu(\beta_1)Z_\nu(\beta_2)|_{F_{l,k}}=\sum_{\mu'\nu'}^{\overline{\epsilon}_\mu+\overline{\epsilon}_\nu=\overline{\epsilon}_{\mu'}+\overline{\epsilon}_{\nu'}}Z_{\mu'}(\beta_2)Z_{\nu'}(\beta_1)|_{F_{l,k}}W\left(\begin{array}{ll}k+\overline{\epsilon}_\mu+\overline{\epsilon}_\nu & k+\overline{\epsilon}_{\nu'} \\ k+\overline{\epsilon}_\nu & k\end{array}|\beta_1-\beta_2\right) \tag{63}$$

and

$$W'\left(\begin{array}{ll}l+2\overline{\epsilon}_\mu & l+\overline{\epsilon}_\mu \\ l+\overline{\epsilon}_\mu & l\end{array}|\beta\right)=r(\beta)$$

$$W'\left(\begin{array}{ll}l+\overline{\epsilon}_\mu+\overline{\epsilon}_\nu & l+\overline{\epsilon}_\nu \\ l+\overline{\epsilon}_\nu & l\end{array}|\beta\right)=r(\beta)\frac{sin\frac{\pi}{\xi}sin\pi(\frac{i\beta}{\hbar\xi}+\frac{l_{\mu\nu}}{\xi})}{sin\pi(\frac{l_{\mu\nu}}{\xi})sin\pi(\frac{i\beta}{\hbar\xi}+\frac{1}{\xi})}$$

$$W'\left(\begin{array}{ll}l+\overline{\epsilon}_\mu+\overline{\epsilon}_\nu & l+\overline{\epsilon}_\mu \\ l+\overline{\epsilon}_\nu & l\end{array}|\beta\right)=r(\beta)\frac{sin\pi(\frac{i\beta}{\hbar\xi})sin\pi(-\frac{1}{\xi}+\frac{l_{\mu\nu}}{\xi})}{sin\pi(\frac{l_{\mu\nu}}{\xi})sin\pi(\frac{i\beta}{\hbar\xi}+\frac{1}{\xi})}$$

$$W\left(\begin{array}{ll}k+2\overline{\epsilon}_\mu & k+\overline{\epsilon}_\mu \\ k+\overline{\epsilon}_\mu & k\end{array}|\beta\right)=r'(\beta)$$

$$W\left(\begin{array}{ll}k+\overline{\epsilon}_\mu+\overline{\epsilon}_\nu & k+\overline{\epsilon}_\nu \\ k+\overline{\epsilon}_\nu & k\end{array}|\beta\right)=r'(\beta)\frac{sin\frac{\pi}{1+\xi}sin\pi(\frac{i\beta}{\hbar(1+\xi)}-\frac{k_{\mu\nu}}{1+\xi})}{sin\pi(\frac{k_{\mu\nu}}{1+\xi})sin\pi(\frac{i\beta}{\hbar(1+\xi)}-\frac{1}{1+\xi})}$$

$$W\left(\begin{array}{cc} k+\bar{\epsilon}_\mu+\bar{\epsilon}_\nu & k+\bar{\epsilon}_\mu \\ k+\bar{\epsilon}_\nu & k \end{array}|\beta\right)=r'(\beta)\frac{sin\pi(\frac{i\beta}{\hbar(1+\xi)})sin\pi(-\frac{1}{1+\xi}+\frac{k_{\mu\nu}}{1+\xi})}{sin\pi(\frac{k_{\mu\nu}}{1+\xi})sin\pi(\frac{i\beta}{\hbar(1+\xi)}-\frac{1}{1+\xi})}$$

It can be check that the Boltzmann weights $W\left(\begin{array}{cc} c & d \\ b & a \end{array}|\beta\right)$ and $W\left(\begin{array}{cc} c & d \\ b & a \end{array}|\beta\right)$ satisfy the star-triangle equations (or Yang-Baxter equation)

$$\begin{aligned}\sum_g \quad & W\left(\begin{array}{cc} d & e \\ c & g \end{array}|\beta_1\right)W\left(\begin{array}{cc} c & g \\ b & a \end{array}|\beta_2\right)W\left(\begin{array}{cc} e & f \\ g & a \end{array}|\beta_1-\beta_2\right) \\ & =\sum_g W\left(\begin{array}{cc} g & f \\ b & a \end{array}|\beta_1\right)W\left(\begin{array}{cc} d & e \\ g & f \end{array}|\beta_2\right)W\left(\begin{array}{cc} d & g \\ c & b \end{array}|\beta_1-\beta_2\right) \end{aligned} \tag{64}$$

and unitary relation

$$\sum_g W\left(\begin{array}{cc} c & g \\ b & a \end{array}|-\beta\right)W\left(\begin{array}{cc} c & d \\ g & a \end{array}|\beta\right)=\delta_{bd} \tag{65}$$

Actually ,the Yang-Baxter equation Eq.(64) and the unitary relation Eq.(65) are direct results of the exchange relation of vertex operators in Eq.(62) and Eq.(63) (the associativity of algebra $Z_\mu(\beta)$ and $Z'_\mu(\beta)$)

Remark: In fact ,we have construct the bosonization of the vertex operators for the trigonometric SOS model of $A^{(1)}_{N-1}$ type.

Discussions

We have constructed a $\hbar$-deformed W_N algebra and its quantum Miura transformation.The $\hbar$-deformation of W_N algebra can be obtained by two way: one can first derive the classical version of $\hbar$-deformed W_N algebra from studying the Yangian double with center $DY(\hat{sl}_N)$ at the critical level as E.Frenkel and N.Reshetikin in studying the $U_q(\hat{sl}_N)$ at critical level [10], then one construct the (quantum) $\hbar$-deformed W_N algebra by quantizing the classical one ; Another way to construct the $\hbar$-deformed W_N algebra is by taking some scaling limit of q-deformed W_N algebra like Eq.(16). Actually, the same phenomena also occur in studying Yangian double with center $DY(\hat{sl}_N)$:the $DY(\hat{sl}_N)$ can be considered as some scaling limit of $U_q(\hat{sl}_N)$ algebra.

We only consider the $\hbar$-deformed W_N algebra for generic ξ .When ξ is some rational number ($\xi=\frac{p}{q}$,p and q are two coprime integers), the realization of $\hbar$-deformed W_N algebra in the Fock space $F_{l,k}$ would be highly reducible and we have to throw out some states from the Fock space $F_{l,k}$ to obtain the irreducible component $H_{l,k}$. (Here we choose the same symbol as that in [1]). For N=2 ,the irreducible space $H_{l,k}$ (i.e $L_{l,k}$ in [12]) can be obtained by some BRST cohomology [12]. Unfortunately,the constructure of the BRST complex and the caculation of cohomology for $3\leq N$ is still an open problem .

We also construct the vertex operators of type I and type II. These vertex operators satisfy some Fadeev-Zamolodchikov algebra with the face type Boltzmann weight as its constructure constant.In order to obtain the correlation functions as the traces of products of these vertex operators, we should introduce a boost operators H_1

$$H_1 = \int_0^\infty -\frac{tsh\frac{N\hbar t}{2}sh\frac{\xi\hbar t}{2}}{sh\frac{(N-1)\hbar t}{2}sh\frac{(1+\xi)\hbar t}{2}} a_1(-t)a_1(t)dt \tag{66}$$

which enjoys the property

$$e^{xH_1}a_1(t)e^{-xH_1} = e^{-xt}a_1(t)$$

Morever,using the skew-symmetric fusion of N's vertex operators ,we can obtain some invertibility for our vertex operators of the form like (3.19) and (c.20) in Ref.[1]. Then the correlation function can be described by the following trace function

$$G(\beta_1, ..., \beta_{Nn})_{\mu_1,....,\mu_{Nn}} = \frac{tr(e^{-xH_1}Z'_{\mu_1}(\beta_1)....Z'_{\mu_{Nn}}(\beta_{Nn}))}{tr(e^{-xH_1})} \tag{67}$$

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